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Growth in Students' Conceptions of Mathematical Induction

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ABSTRACT

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While proof and reasoning lie at the core of mathematical practice, how students learn to reason formally and build convincing proofs continues to invite reflection and discussion. To add to this discussion I investigated how three students grew in their conceptions of mathematical induction. While each of the students in the study had different experiences and grew in different ways, the grounded axes (*triggering events, personal questions about mathematics,* and *personal questions about a particular solution*) highlighted patterns in the narratives and from these patterns a theoretical perspective emerged. Reflection, both on mathematics in general and about specific problems, was central to students' growth. The personal reflections of students and triggering events influenced each other in the following way. The questions students wondered about impacted which trigger stimulated growth, while triggers caused students to rethink assumptions and reflect on mathematics or specific problems. The reflections that allowed triggers to stimulate growth along with the reflections that were results of triggering events constitute an "investigative orientation." Each narrative reflects a different investigative orientation motivated by different personal needs. These investigative orientations affected what type of knowledge was constructed.

Keywords: Mathematical Induction, Proof, Undergraduate students

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The thesis of John Gruver is acceptable in its final form including (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory and ready for submission.

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Chapter 1: Introduction

> "Proof is what makes mathematics special. Students should understand and appreciate the core of mathematical culture: the value and validity of careful reasoning, precise definition, and close argument." (Mathematical Association of America, 2009, p. 6).

While proof and reasoning lie at the core of mathematical practice, how students learn to reason formally and build convincing proofs continues to invite reflection and discussion. To further this discussion, it may help to investigate, in detail, how specific students learn to build correct induction proofs with understanding. On this basis, this study seeks to offer insight into how educators might help undergraduates learn to reason mathematically.

The term "meaningful induction proof" could have a variety of interpretations. Here, a proof will be an argument that can be accepted in the professional mathematics community. We will say a proof is meaningful if it is meaningful to its author, in the sense that it is convincing to him or her in the same way that it is convincing to the mathematics community. In particular, the author does not simply follow a prescribed template to gain approval from an external authority, but instead constructs an argument that makes sense and is compelling personally. Students who build and check such proofs participate in genuine mathematics, mathematician's mathematics, because at that moment they no longer experience mathematics merely as spectators.

Proof by mathematical induction, at least for beginners, might be especially susceptible to a template approach. When mathematical induction was first introduced to me, I understood it only as a template to be filled in. I could "do" step one and step two, but the resulting proofs were not meaningful, in the sense defined above, because I didn't understand why my proofs proved anything. Yet my teachers seemed content with the "proofs" I had produced. Later, as I began to understand the logical structure of proofs by mathematical induction, such proofs became more than just exercises; they became meaningful. The contrast was stark. One year I was just performing a procedure; mysteriously, a few years later, I could explain the logical structure of induction proofs. As prior research has confirmed, (Brown, 2003; Fischbein & Engel, 1989; Movshovitz-Hadar, 1993b) many students conceptualize mathematical induction simply as a series of steps. In other words, my own case might represent a larger population. It would benefit undergraduates who view induction in this way to make the transition from filling in a template to constructing a meaningful argument.

Despite some prior research, this transition remains mysterious. While the literature on mathematical induction offers useful categorizations of what induction means to students, and documents student deficiencies, the mathematics education community still lacks detailed analysis of how students can come to understand induction meaningfully. I studied people who grew in their conception of induction. Although none of the subjects started with a template understanding, their conceptions of induction were enriched through the course of the study. Detailed analysis of student thinking at key transition points offered insight into how students' understandings of induction grow.

Three initial questions focused the research: a) how do students build meaningful conceptions of induction; b) how does the teacher facilitate this construction of knowledge; and c) what aspects of the mathematics need highlighting to ensure this knowledge construction? We should note immediately that these questions represent simply initial guidelines for investigation. In particular, the research design, as it unfolded, afforded opportunities for critical reflection, and refinement of these focal questions, based on the emerging student data. In this way, both data gathering (primarily through ongoing student interviews) and analysis proceeded in iterative cycles, to reflect emerging insight into student thinking.

Chapter 2: Literature Review

Prior research on teaching and learning mathematical induction seems to fall into three broad categories: stage theories, pedagogical suggestions, and descriptions of student difficulties.

The stage theory literature is represented by Brown (2003), Dubinsky and Lewin (1986), Harel (2001), and Harel and Brown (2008). I will emphasize Harel and Brown (2008), which is a summary of Harel (2001) and Brown (2003). The stages advanced in these papers seem more meaningful than those put forward by Dubinsky and Lewin (1986). The result of Harel and Brown's (2008) synthesis is a three-stage model of learning mathematical induction. In the first stage, students "show" a theorem is true by showing that the theorem holds for a few cases. In stage two, students begin to see a pattern in what mathematicians call the induction step. However, they do not prove the induction step in general terms. Rather, they show that it is true for a few examples, (e.g. $P(1) \Rightarrow P(2)$ and $P(2) \Rightarrow P(3)$ so the pattern must continue). In stage three, students actually create proofs by mathematical induction. Most importantly, students know $P(n) \Rightarrow P(n+1)$ because they have proved it deductively, not simply because a few examples, for small n, worked.

Even though Harel and Brown's (2008) stages are useful categorizations, they do not get at the details of how a student progresses from one stage to another, or more generally what elicited the growth in students' conceptions. Harel might claim that the curriculum design enabled the transition, but this is an incomplete explanation. To understand why students are able to grow in their understanding, a deep analysis of changes in students' conceptions of induction needs to be performed. The purpose of the present study is not to categorize all stages a learner must pass through to understand induction, but to understand the experiences that help a student's conception grow. As the participants' conceptions grew, the details of how this growth happened for these students were captured by paying close attention to how the task and teacher influenced the growth.

Suggestions on teaching induction are advanced by Avital and Hansen (1976), Dubinsky (1986, 1989), Leron and Zazkis (1986), Movshovitz-Hadar (1993a) Wistedt and Brattström (2005), and Woodall (1981). I will highlight Avital and Hansen (1976) and Wistedt and Brattström (2005), because these two articles together discuss principles of teaching that were relevant to this proposed study. Avital and Hansen (1976) strongly emphasized the importance of student exploration. They noted that in textbooks students are most often asked to use mathematical induction to prove a specific equality. For Avital and Hansen, the nature of such a task is not consistent with the full experience of a mathematician in creating a proof. Mathematical researchers engage in exploration to imagine a plausible result, and then they seek to provide a proof of the result. Simply asking for a proof of a pre-formed result without offering a chance to think about and perhaps discover the result is not genuine mathematics. Teachers should offer students the opportunity to explore before producing the final product, the proved theorem.

Even though students' exploration is important, teachers play a critical role in helping students understand. Wistedt and Brattström (2005) observed students working collaboratively on a non-standard induction problem, but this process did not result in growth of students' conceptions. Wistedt and Brattström conjectured that this was because none of the students understood induction, so they could not teach each other. This study showed that simply putting peers into groups and having them discuss mathematical topics is not sufficient for learning to take place.

Avital and Hansen (1976) and Wistedt and Brattström (2005) together provide insight into how students' roles and teachers' roles might be defined in ways that would help students understand how to create induction proofs. Students need to be actively engaged in exploring concepts, making conjectures, reformulating conjectures, and proving them. However, it is not enough to give the students a rich problem and leave everything up to them. An insightful teacher who asks questions that draw students' attention to previously unexamined aspects of a problem must support students.

Documenting the difficulties students have with induction is a common theme in much of the literature on mathematical induction (see Ernest, 1984; Fischbein & Engel, 1989; Lowenthal & Eisenberg, 1992). Harel (2001) showed that often the difficulties students have depend on how they were taught. While Harel's observation might be seen as qualifying other results cited here, a reoccurring theme emerged, the formulation of and argument for the induction step was problematic for students (Ernest, 1984; Fischbein & Engel, 1989; Lowenthal & Eisenberg, 1992). While this issue was considered in the present study, the students in this study did not seem to have as much difficulty with the induction step as was reported in the literature.

Chapter 3: Theoretical Orientation

Here I describe two analytic lenses that helped focus my analysis, which led me to segments of the data where students thought about proofs as a mathematician does.

For mathematicians, proof is a complex process of purposeful exploration and discovery. The exploration that accompanies proof engenders understanding, which is what mathematicians find valuable (Hanna, 1983). Reid (1996, p. 186) said, "proving is central to mathematical discovery and exploration." Hence, data segments that give evidence of students' use of proof to drive exploration merit careful analysis. Raman (2003) and Lakatos (1976) provided constructs to help locate and conceptualize those instances in the data where students are thinking about proofs like mathematicians. Such instances were the starting points for an analysis, which evolved over time.

Raman (2003) claimed that for mathematicians, proof was essentially about "key ideas." A key idea "links together the public and private domains, and in doing so gives a sense of understanding and conviction. Key ideas show why a particular claim is true" (Raman, 2003, p. 323). The private aspects are the parts of the proof that explain or enlighten the individual. The public aspects are the parts that make the argument convincing or sufficiently rigorous to the mathematical community. In interviews I identified what the key ideas appeared to be for the students, and concentrated analysis at places in the data where students reference these ideas.

I also looked for places where students refined ideas by proposing conjectures or "solutions" and then either prove or refute them. In his innovative book, *Proofs and Refutations*, Lakatos (1976) showed that a critique of a proposed proof can be a motivating force for the development of new mathematical exploration and knowledge. In the situation presented in the book, the refinement happened because individuals challenged one another's justifications. Such refinement can happen in a classroom as well, the place where Lakatos, indeed, has set his fictional retelling of the historical story. Such challenges or interplays could happen within an individual or socially within the classroom. As I reviewed the data, I looked for evidence of such interplays in the interviews.

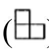
Chapter 4: Research Subjects, Setting, and Task Design

The subjects of the proposed study were undergraduates in Math 290. This course was designed to introduce students to mathematical proof and increase their ability to communicate mathematically ("09 – 10 undergraduate catalog," 2009). I selected participants from one section, led by an instructor sympathetic to the goals of this study. Participants were selected to represent the broadest student population possible. My goal was to have racial diversity, a variety of working styles, a range of class standings, and several areas of the country represented.

In contrast to the course, my goal was to gain insight into the growth of students' concepts of induction through investigation of their thinking at a key developmental threshold. My guiding questions deal with how students work and think as they wrestle with key ideas. Accordingly, the participants were selected from students who already have some acquaintance with induction.

Subjects participated in three settings: an initial individual task-based interview, a small group session, and a follow-up individual interview. In the first interview I documented students' initial concepts of induction. Data collected here provided data to compare with data collected later in the study. In the group session I asked students to solve two problems in collaboration, to elicit mathematical discussion with minimal researcher input. In this way, students' conceptions of induction could develop, or at least become more visible. By putting students in a group setting where they could collaborate, I especially hoped for opportunities to see how ideas were refined through a social process of confirming or refuting proposed solutions, conjectures, and guesses. Individual interviews, before and after the collaborative session, were used to document to what extent students' conceptions change. The follow-up interviews not only investigated whether growth had occurred, but also allowed me to identify possible key ideas. Whenever key ideas appeared, especially in the second round of interviews, I invited the students to explore them on the spot.

For productive collaboration, the task the students undertook should challenge them to reflect on basic understandings. If students were to propose key ideas, the task needed to be rich enough to elicit or require such ideas. Also, for students to refine their thinking through discussion, the task needs to be rich enough, or pose sufficient challenges, to demand discussion. For specific tasks, I chose problems with these two requirements in mind.

The modified chessboard problem asks students for which n a 2^n x 2^n chessboard with one square removed can be tiled with L-shaped () pieces, regardless of which square was removed. The handshake problem asks students to determine how many hands a hostess at a party shakes, given certain constraints. The party consists of 2n partiers and each partier, excluding the host, shakes a different number of hands. Also, no one shakes the hand of his or her spouse. Since these problems were unlike problems they had faced in their class, students needed to discuss with group-mates how to solve them. Also, solutions of these problems tend to have explanatory power that goes well beyond whatever templates might have been in place initially. Since I limited the number of participants, I could analyze particular events in detail to locate analytic constructs with explanatory power in relation to my guiding questions.

Chapter 5: Data Collection and Analysis

Data Collection

I collected data in four stages: a preliminary questionnaire, a small-group session devoted to a single task, and two rounds of individual interviews, before and after the group session. The interviews and the group session were videotaped with one camera, to capture the participating students and their work. Key events were selected and transcribed for subsequent analysis.

At the outset, everyone in the class who consented to participate completed a brief questionnaire. The questionnaire asked only for basic information such as name and year at BYU, prior experience with induction, and if so, a very brief description of the latter. I also observed the class, and took field notes to better understand students' capabilities and working styles.

The next step was an open-ended task-based interview with each of the students, one-to-one. The tasks were unassigned induction problems from their text, more basic than the modified chessboard or the handshake problem. I asked questions like, "How is this argument convincing?" or "Could you tell me more about your thinking here?" or "Are you sure?" In these interviews I worked primarily to encourage each student's exploration, to convey interest and curiosity about their thinking, and to provide generous opportunities for students to develop explanations, locate errors (when such occurred), and for the students to reshape their arguments and strategies as needed. The best interviews of this kind occur when the student subject takes over the conversation and actively explores new possibilities.

For the group session, the students met together for about two and a half hours, to work collaboratively on the modified chessboard and handshake tasks. The students working together were videotaped by a single camera, which zoomed in to capture student work when appropriate.

The final interview with each of the students had two main components. First, I showed the students clips from their initial interview and asked them to characterize their earlier work seen in the video clips. This offered me both a clearer understanding of the students' current conception as well as starting places to think about what changes were significant for them. The second component centered on the progress each student made toward a solution to the two tasks as a springboard to how their conception of induction had evolved. The students' individual progress reports offered them opportunities to explain their ideas about the two tasks specifically and about induction more generally.

Analysis

Analysis was a variant of grounded theory. First, I did a coarse descriptive analysis, describing, at 10-minute intervals, what happened in the videotaped session at that moment. From this broad survey, I used the guiding theoretical constructs previously described to identify key events in the data. I also included events and ideas that students identified as significant. I began a detailed analysis of these key events by refining the descriptive analysis around the event to 30-second intervals. This note taking was done with as little interpretation as possible. During this open coding portion of analysis, I looked for emergent themes. From these themes I established analytic axes, in the style of grounded theory (Strauss & Corbin, 1990) to help focus further coding and analysis, especially to make clear, as much as possible, the dynamics of the learning process. In the process, I took repeated passes through the data, revising as necessary, to make sure, axis by axis, that my interpretations fit the data and provided insight into the dynamic that I sought to clarify. The three axes I eventually selected are descriptive in nature. They are (1) personal questions about mathematics, (2) personal questions about a particular solution, and (3) triggering events. Basing analysis on these three axes helped me make clear, to a significant extent, the dynamic of these students' learning process. In particular, I found that the personal questions students had influenced which triggering events would stimulate their growth. Similarly, a triggering event could stimulate reflection and create new personal questions. To illustrate these findings in more detail I now present data segments and analysis of these segments using the thematic axes.

Chapter 6: Background, Data and Analysis

Analysis was focused on events that illustrate growth in students' conceptions of induction and the context surrounding these events, with the intent to learn why the students were successful in understanding induction better. During the course of the project, students' conceptions of induction grew in different ways, so events were selected for each student to reflect these differences. To underpin analysis, the context surrounding each event was analyzed. This context can be understood in terms of the axes described previously. First, there were events that stimulated growth. I call such stimuli *triggering events*. Sometimes the trigger is a single event and sometimes it is a cluster of events. Second, there are circumstances that affect the way a trigger actually stimulated the student's growth. Such circumstances include both student's *personal questions about mathematics* and *personal questions about a particular solution*, as well as, that student's working style and personal motivations.

To understand the data, it is first important to understand each student's background and the events they experienced which helped them grow. First, I summarize relevant background that is common for all students. Then I present a grounded narrative for each student's event, including further background specific to that student. I conclude with themes and patterns that emerge directly from the data in relation to my guiding questions.

Background

The project had three sections: an initial interview, a group session, and a final interview. The initial interview included giving students four tasks that could be solved by induction. These tasks were to prove: (1) $n! > 2^n$ for n greater than or equal to 4, (2) the sum of the first n odd integers is n^2, (3) the finite intersection of complements is equal to the complement of the union, and (4) every non-empty finite subset of the real numbers has a largest element. In the group session, the students were asked to solve two problems: first, to determine which chessboards, in terms of size, with one square removed can be tiled with L-shaped tiles, and, second, to determine how many hands a hostess shook at a party, given certain constraints. During the final interview, I showed each student clips from his or her first interview and asked the student if he or she would respond differently. I also had the student recapitulate his or her solution to the two problems from the group session.

During the group session, the students tried many ideas to solve the chessboard problem, but eventually gave the following argument. They began with a proof of the 2 x 2 case. They then assumed chessboards of size 2^n x 2^n could be tiled and investigated chessboards sized 2^{n+1} x 2^{n+1}. They divided the chessboard into four quadrants. They argued that the quadrant with the missing tile could be tiled by the induction hypothesis. This leaves a large "L" (it is actually a backwards L, but I will refer to both L's and backwards L's as L's) consisting of the other three 2^n x 2^n quadrants. To do this they first broke this "L" into four smaller L's (see Figure 1). Alexander pointed out that each of these four L's is basically the 2^n x 2^n case, with one quadrant removed. The group claimed this implies each of the L's can be tiled (see Figure 2). Their solution leaves the following question unanswered. How does one know that the small L's can be tiled? The students seem to have thought that because it was possible to tile the 2^n x 2^n board with one square removed, it was also possible to tile the board with one quadrant removed. This question was discussed individually in the exit interviews. Reflection on this question was often a way for students to understand the solution to this problem better and with Mark, induction itself better.

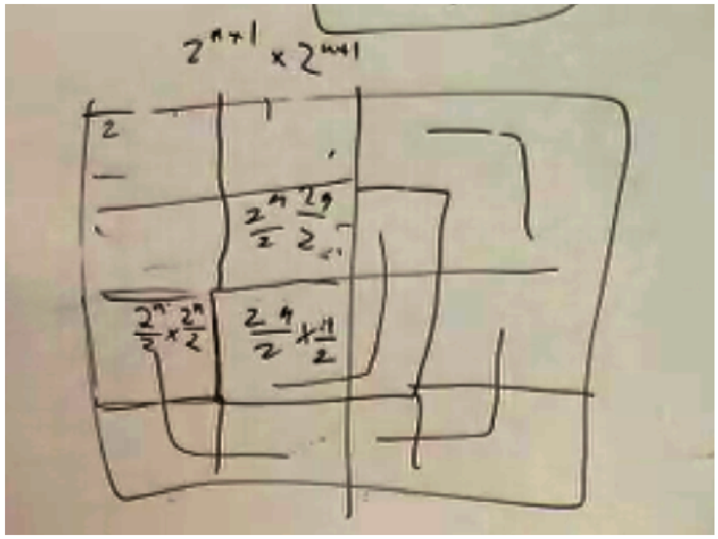


Figure 1. The 2^{n+1} by 2^{n+1} case subdivided into the 2^n by 2^n case and four “L’s.”

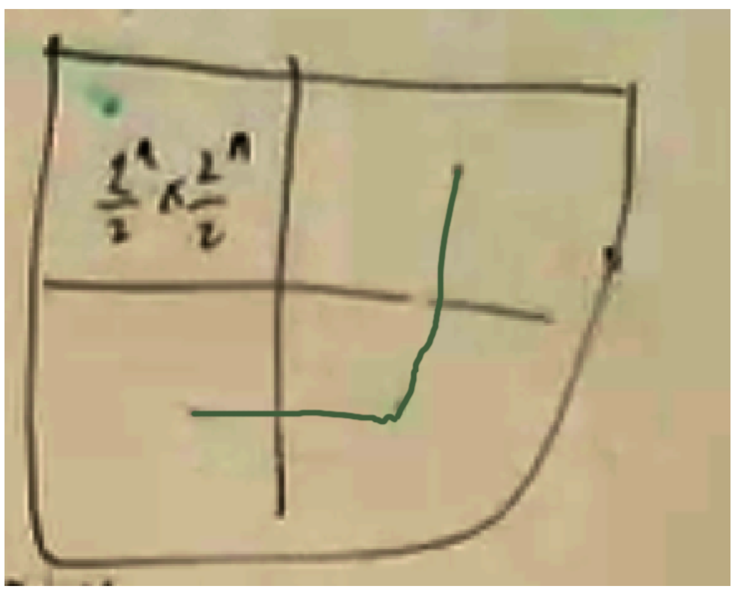


Figure 2. The 2^n by 2^n case, highlighting a backwards L.

Mark

Introduction. Mark came from a large family of seven, which resided in a small town in Pennsylvania. Mark was a junior, a mechanical engineering major, and a math minor. He enrolled in math 290 to fulfill a requirement for his minor. Previous to math 290 he had taken courses in calculus I and II, multivariable calculus, linear algebra, and differential equations. He said he loves math and compared solving problems in the initial interview to going to an amusement park.

Initial Interview. In Mark's initial interview, he mentioned that induction was unclear to him at first introduction, but coming into the project he understood it well. He reported induction became clearer for him after he had solved several homework problems. In the initial interview he was able to do each of the problems I gave him by induction. More than that, he was able to explain why these proof proved the proposition for all n.

One important feature of Mark's problem solving style is that, after he finished an argument, he reflected on the clarity and validity of his solution. For example, he reported, without prompting, that he could not find a "neat" way of presenting his proof that $n! > 2^n$ for n greater than or equal to four. Also, when he proved De Morgan's law for an arbitrary finite number of sets, he commented that although his argument made sense to him, he did not think it would stand up in a mathematical journal.

The De Morgan's law event is an example where Mark's ability to articulate his solution grew. After he had reflected and said he did not think his proof would stand up in mathematical journal, I asked what part of the proof a mathematician would criticize. He answered quickly. After he articulated the problem, he saw a clearer way to prove the law and volunteered his improved solution. Here the *triggering event* was a question. Mark was not satisfied with his solution, and with some encouragement from me, he decided to rethink the proof and then came up with a better solution. While the triggering event was me asking Mark to analyze his solution, the context of Mark's *personal question about a particular solution,* in this case a question about his own proof, as well as, his working style were more important. I believe it is his personal question that allowed the triggering event to actually stimulate growth.

Exit Interview. In his exit interview, I asked Mark to give his solution to the modified chessboard problem. His solution was based on an idea that had been mentioned in the group session. The solution was similar to the final argument given in the group session, but not the same. He broke the chessboard into four quadrants and considered the three of them without the missing block, the three forming an L. He then broke this L into smaller L’s using the same procedure as used in the group session. He then said he would break these smaller L’s into even smaller L’s, using the same procedure (see Figure 3). Mark explained, you continue this process until you have L’s that are made up of three 1x1 squares. This is your tiling. To tile the quadrant with the square removed, you again break the quadrant up into four quadrants; the three without the missing square form an L. You can tile this L using the same process as above. You continue this process.

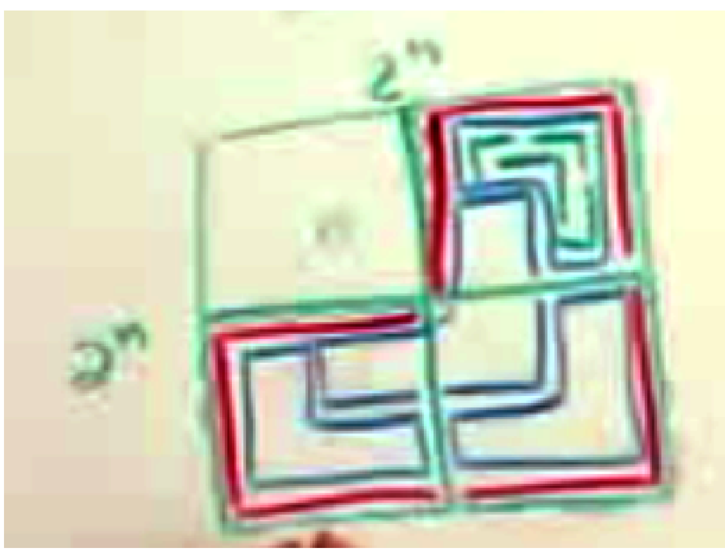


Figure 3. Mark breaks down a red L into four blue L’s and a blue L into four green L’s.

At this point in the interview, Mark had come up with a solution to the problem, but I was not satisfied with the precision of his solution. What follows next in the interview is my expression of dissatisfaction and Mark's response. Here we again see a triggering event.

Bracketed items are not part of the original transcript.

> John: Um… So…. Like, I guess, this like seems like an induction argument sort of, but it's like, um, not the traditional induction argument right? And so I guess, um, for, for me, uh, like… When you do induction, like, one, one way you could do it, is just say, like, see this is how I get from one to two, and then this is how I get from two to three, and this is how I get from three to four and then kind of being like and this will still continue (*simultaneously* Mark: continue), right? But when we do like a formal mathematical induction argument we like really nail down, like, how you're gonna get from one case to the next. (Mark: Ok) And so like, in this, we're, we're breaking things down, so instead of building up, we're breaking down, umm, but we're still kind of being like, like, to, the
>
> (interrupts) Mark: Vague
>
> John: Yeah, a little vague. Like, the, the reader of the proof, or whatever, has to do like a lot of imagining. You know what I mean?
> [My question again acts as a triggering event]
>
> Mark: Yes
>
> John: So, is there a way we could, um, you know, sharpen it up a, a tiny bit?
>
> Mark: I think it just clicked in my mind how to do it. (John: Okay. Okay.) Let's go for. And. So if we have a 2n by, (Mark draws a square on the board) or 2 to the n plus one, so we're going to assume it works for the 2 to the n case. (John: okay) And then we go to the 2 to the n plus one case, we've already shown it equals, it works for n equals one. (John: K.) So this is 2 to the n plus one by 2 to the n plus one. (Mark labels the sides of the box he has drawn) (John: K.) And if we divide it, like we've down before. (Mark divides the box into four quadrants). (John: K.) Then we get blocks that are 2 to the n by 2 to the n. (Labels the upper right quadrant 2^n on both the top and side) (John: okay.) Now if we look at the block that has been, that has had the block removed. (John: uh-huh.) So, let's say it's this one (Draws a small square in the upper left quadrant). (John: K.) We know that it works for 2 to the n. (John: Okay) And so we know that we can tile this block (Traces the upper left quadrant with his finger) (John: okay.), with umm, (Pause 3 seconds) the tiles. (John: uh-huh.) And for the rest of this (Gestures to the other three quadrants) (Pause 6 seconds)… Then (pause 16 seconds)… Ah. I just had a good idea.
>
> (Excited) John: Okay, what is that?
>
> Mark: We know, we know we can tile these 2 to the n blocks if one tile has been removed (Places fingers on the upper right quadrant). (John: K.) And for this, this

> three corners case, then let us remove the block on the very corner here, from all (draws a small square in the lower left corner of the upper right quadrant, a small square in the upper left corner of the lower right quadrant, and a small square in the upper right corner of the lower left quadrant,)… all of these blocks (Gestures to the three quadrants) (John: Okay). And then, we know we can tile this one now (Points to the upper right quadrant), since one has been removed (Points to the small square in the upper right quadrant). We know we can tile this one (Points to the lower right quadrant), since one has been removed (Points to the small square in the lower right quadrant). And we know that we can tile this one (Points to the lower left quadrant), since one has been removed (Points to the small square in the lower left quadrant). And then we're just gonna take the one single one (Points to a previously drawn L shaped tile) and stick it right there in the corner (Shades the three adjacent small squares) to complete those three (Gestures to the three quadrants). (John: Ok) And so then it works for the 2 to the n plus one case, which proves that.

Here Mark went through the process of making his argument more clear and in the process he revised the argument substantially. Instead of proving an arbitrary n case directly, he used induction. Instead of breaking down the n case, he makes use of previous cases. One salient feature of his argument that changed is that in order to prove the n+1 case, he used two different tilings from his induction hypothesis. One where the block has been removed in the same place that it has been removed in the upper left quadrant, and one case where it has been removed in a corner. His induction hypothesis had to apply not only to one object, but to a whole class of objects. In other words, he had to assume the hypothesis true for *all* possible squares removed. This is how he could use two different tilings from his hypothesis.

Discussion. Harel's approach (2008) highlights the significance of Mark's achievement. In his paper, Harel (2008, p. 122) distinguished between problems that require the solver to assume the hypothesis for one object versus those that require the assumption for a whole class of objects. He suggested a teacher start by giving problems of the former type, where the induction hypothesis need only apply to an object, then move to the more complex type of problem. This helps the student move through the stages he described, first placing emphasis on recursive thinking and then giving a greater intellectual need for reliance on the induction hypothesis.

In the exchanged relayed above, Mark seems to have transitioned to a new way of thinking about induction, in the sense that he gained the ability to think of the induction hypothesis applying to a whole class of objects. Mark was able to make this transition because he thought about the chessboard problem in a new way. He gave an argument and then I asked him to make it clearer. This triggered Mark to approach the problem from a new angle. Mark's fresh look helped him not only solve the problem, but also see induction itself in a new way. His in-the-moment reflection about his solution was central to his growth in understanding. Also, it seems he was not completely convinced of his original solution. When I tried to articulate my concern with his proof, he finished my sentence with the adjective vague. It did not take much convincing to get Mark to improve his proof. This makes sense considering his working style of proving and then reflecting on the validity of the proof.

As with the De Morgan's law example, Mark's *personal questions about his solution* helped him grow in his understanding. However, this time the growth was more substantive because it is growth in Mark's conception of induction itself. In order to make his solution clearer and more concise, he revised how he thought about induction itself.

Sarah

Introduction. Sarah was a sophomore from New Hampshire. She was majoring in actuarial science with a math minor. She had taken several math courses: linear algebra, Calculus I and II, and multivariable Calculus. She was involved with IMPACT, a faculty advised research group for undergraduates. During the first interview, I asked if she had seen induction in IMPACT. She said she had, but did not understand induction when first introduced. She said that she had only mimicked what others had done. We talked about how her conception had changed since then. The transcript of this exchange will be presented later.

Initial Interview. During the initial interview, Sarah attempted proofs of the following three results: the formula of the sum of the first n odd integers, generalized De Morgan's law, and finite subsets of the real numbers have a largest element. The first two she was able to do, but she was not able to come up with a proof for the third. Two interesting features came up in the initial interview. First, Sarah's difficulties centered on technical details and novel problem situations, not on induction itself. Second, an important part of induction for Sarah seemed to be building a particular set at the beginning of each proof.

The first feature of her problem solving process was that she did not have any difficulty with the induction part of the proofs, just with technical details and novel situations. In the odd integers problem she had trouble with some of the symbolic manipulations. However, she was able to explain how induction proved the formula for all n. The hardest part for Sarah of proving De Morgan's law was the proof of the law for two sets. Proving the law for k+1 sets assuming it holds for k sets, the induction step, posed no problem to her, even though she said she had never done a problem like it before. When attempting the largest element problem, she mentioned several times that she did not have experience proving these types of problems by induction.

The second feature was that when she tried the induction problems, she would first write out, using set builder notation, a subset of the natural numbers that had the property she was trying to prove applied to all natural numbers. This is how her instructor introduced the concept to the class. First one writes this set, then one proves the set contains all the natural numbers. Her consistency in giving this preamble to the proof is evidence that she saw this as important to the proof. Later, this became less important for her.

Exit Interview. Sarah's growth in her conception of induction occurred in when induction could be applied. This was revealed in the exit interview when she watched and responded to a clip from her initial interview. For Sarah, the *trigger* was not a single event, as it was for Mark. Rather her growth was triggered by exposure to rich induction problems.

In the clip from the initial interview Sarah watched, she basically described her understanding of induction. I showed her this clip because I wanted to see if she would describe things differently after her new experiences with induction. While she said her understanding was mostly the same, she mentioned her view of when induction could be used had broadened. First I present the transcript of the clip she is responding to, her response to the clip follows.

> *Clip from interview one*
> Sarah: 'Cause like, for IMPACT, we did, like I would do a proof and it would show, like I didn't get why, like, you assume it's true for, like, one or something, and then you do it for, so you assume it's true for some random thing, and then you show that the next one would be true, and that didn't make sense to me how that proved why everything was then true, the whole like sequence or whatever you were proving. So that didn't make sense to me. But then doing it in class. I guess, when we just, he started doing more examples, is showing that it is like a sequence that we were proving or something like that. (John: Um-hmm) And then you'd show it's true for one. And then, you'd show like for some random k, and then for the next one. And that made more sense to me, 'cause then I realized that like, you could apply that to like the base case, and then, show that the next one is true and then the next one and keep going. So, that's what made it click for me.
>
> *Response to Clip*
> Sarah: Okay. Umm . . . . I think [my understanding] it's kind of the same. Except now, like, because of the problems that we did when we all worked together (John: uh-huh), umm, I've seen just more how it can be used (John: Ok), and that makes me understand it better still, I think. 'Cause before I just thought that, I only, like, understood how you would apply it to like a series really I guess. But now, having, having seen it done with other examples, I see how you break it down and then kind of, would have to show by induction, like, with like a series, or something, kind of. I don't know if that was clear.
>
> John: What do you mean by a series or something?

Sarah: So, it's not, you wouldn't necessarily break down the problems that we did into series, or like, but it's, it's a pattern that you would show, (John: ok) and by like the pattern is how you would show, like, you can show it's true by induction.

John: Ok, How does that differ from…

Sarah: From what I thought before? Like before, I always, was stuck, in like, like, uh, like, express it mathematically. Like I, I didn't know how to do it not with just, I don't know, like, hmm…

John: What do you mean by express it mathematically?

Sarah: So, like I would write it all out in symbols. And be like this is the base case and this is the next case, and this the next case, and this is the next case. (John: Ok) But, and I didn't think that you could generalize it to be like, well, think of this as your base case. (John: Uh-huh) Like, I thought the base case would be like for n equals 1 or for the first whatever the thing was defined. I guess it's still is kinda the same, I just hadn't seen it been applied to like more of a pattern than like some kind of series in mathematics.

[We continue our discussion to try to clarify how her understanding has changed. For the full transcript see the appendix. Eventually we proceed with the following.]

John: So I guess, umm, what I'm hearing you say, tell me if I'm right or wrong (Sarah: Ok), is that umm, before there were, there was, umm, maybe some sort of formula that (Sarah: yeah) you (Sarah: right) were trying to prove (Sarah: yeah). So you could kind of uhh . .

Sarah: You just (John: mess . . .) you knew what you were arriving at and you knew what started with and you just kind of get there. Like, it was just systematic like plug this in do this swap this around and then you could get there or something.

John: Yeah, (Sarah: Whereas) Go ahead

Sarah: Whereas, now it's just you have more, you want to see if something is true and so you're, ummm . . . It's like not a formula you're getting to, but you're just showing it would be true in any case.

I think a reason why the change that had just taken place was difficult for Sarah to describe is because her knowledge *of* induction did not change, but her knowledge *about* induction changed. She had added a new dimension to her understanding. While induction is still induction, it can now apply to a much larger class of problems and be used to explore, not just to verify.

Discussion. For Sarah, before participation in the study, induction seemed to be a straightforward game of symbol manipulation. While she understood why induction proved the claim, the technique was limited in scope. She moved beyond this view, broadening her conception of when induction could be applied. During the exit interview, she said that she no longer needed to preface her argument of De Morgan's law with a set describing the subset of natural numbers that had the property law's properties. This provides more evidence, that for Sarah, induction had moved beyond a formal exercise.

Her exposure to the chessboard and other rich induction problems helped her realize that induction can be applied to a variety of problems and that it is not always a straightforward manipulation. This exposure to rich problems along with her attempts to solve them and write up their solutions constitutes the *trigger* in Sarah's case. This *trigger* stimulated *personal reflections on induction*. In particular, Sarah began to think about the variety of situations when induction can be used.

Alexander

Introduction. Alexander was a freshman at BYU studying mathematics. He was from Normal, Illinois. He had taken classes the summer previous to this study, at which time he joined IMPACT, a faculty advised undergraduate research program. Through IMPACT he had been introduced to many mathematical topics, including induction proofs.

Initial Interview. In the initial interview, Alexander was able to do most of the problems. The data collected in this interview gives insight into his understanding of induction and illustrates his desire to use formalities and understand mathematical concepts from a formal perspective. He did the sum of the first odd integers very efficiently, making use of summation notation. He was able to do the inequality problem well. While he did make a small error with the base case, overall his general idea with generalized De Morgan's law was correct. Finally, he did not prove that every finite subset of the real numbers has a largest element, but instead proved if a subset has a largest element it is unique.

Perhaps more important than identifying which problems Alexander could do on his feet is his understanding of induction more generally. When asked directly what his understanding of induction was, he simply explained what one needs to prove in an induction proof. More revealing of his actual understanding are two responses to other questions. First, I asked him if induction problems were convincing to him.

> John: The proof worked out well, but like are induction proofs convincing to you? Like, and if so, or why or why not?
>
> Alexander: Uh, (pause) let's see here.
>
> John: Does, does my question make sense?
>
> Alexander: Yeah, it does make sense. I'm trying, I'm, I'm trying to actually think about your question.
>
> John: Ok
>
> Alexander: Alright. How convincing... Uh… Personally they're the least convincing of all the methods of proof, (John: Ok) because it seems that you're, because you're making this, you're making the k step assumption, where in the, where all the other methods of proof, you, we could prove they work by truth tables, and this is by, uh, I've never seen a proof on why mathematical induction works. Where everything else was proved why it works in class.
>
> John: Ok, um..
>
> Alexander: Except we, we did look at the axioms of real numbers, but I haven't worked too much into that.

I observed the classes on induction that Alexander participated in and I believe that when Alexander said the axioms of real numbers he is referring to Peano's axioms. It seems that Alexander craved a formal argument for why induction works. His inclination for formalization came up in other contexts as well. As mentioned earlier, he chose to use summation notation for the sum of squares proof, something neither of the other students did. Also, while working in the group session, he unexpectedly used the word bijection instead of explaining informally that two sets had the same cardinality, even though the sets were set in a real world context. Also, he chose not to use induction on the greatest element problem because the real numbers do not have a clearly defined successor function. This is the first time successor functions came up with Alexander, a concept that seems to be important for him judging from him bringing up the idea multiple times throughout the interviews. Successors are another abstract, formal idea that gets at the heart of why induction works.

Even though at this point he seems to still be searching for that formalization of induction, he does have an intuitive understanding of why induction works. This is demonstrated by the response that follows. After working on the inequality problem, where the base case is four, I asked him why he could assume the k case in his proof.

> Alexander: Give me a moment… (pause) All right. Uh, We can do it because, uh, since we've proved the minimum case, that it's, uh, necessarily true. Like, because, what we're trying to show is that the next number is, uh, going to follow that same formula. So, when we're, uh, assuming k, we're not really assuming, like there's not really this too much that we're assuming, 'cause the main thing we're trying to prove is that for the next, nat, uh, next, uh, natural number, that it's going to be true. So, if 4, then 5, 6, 7, 8 also. And by assuming this, uh, by assuming the k step, that doesn't really prove anything nor does it add hindrance to it.

Alexander demonstrated that he understood that through repeated use of the induction step and the base case you can get from 4 to whatever number you wish.

Another important point from Alexander's initial interview is that when I asked him if he had any questions about induction, he had one. He wondered why Polya's proof of all horses being the same color was false. This problem did not come up in the Math 290 class that he attended. This shows that he actively *reflected on mathematics* meaning he wondered about mathematical problems that he was exposed to outside the classroom environment and wanted to resolve them.

Exit Interview. In the exit interview, I asked him to show me his solution of the chessboard problem. He proceeded by induction, showed the n=1 case, and then assumed the n case. He drew the n+1 case, and split it up into four quadrants. He tiled the one with the missing block using the induction hypothesis. He looked at the three one-by-one squares that belong to the three corners where the quadrants come together, but abandoned this. He showed that after you remove the quadrant with the missing piece, the L that is left over can be broken into four smaller L's in the same way Mark broke up the chessboard with one quadrant removed. This of course leaves the question, how do you know you can tile those smaller L's. I asked him this and our discussion follows.

> Alexander: Umm.. We… I was… umm… by induction. Because, if we have, uhh, we can do it for, ok so, here's a base case. (Points to a previously drawn figure of a tiled 4x4 chessboard)
>
> John: Where?
>
> (Alexander erases a quadrant from the figure, leaving behind Figure 4.)

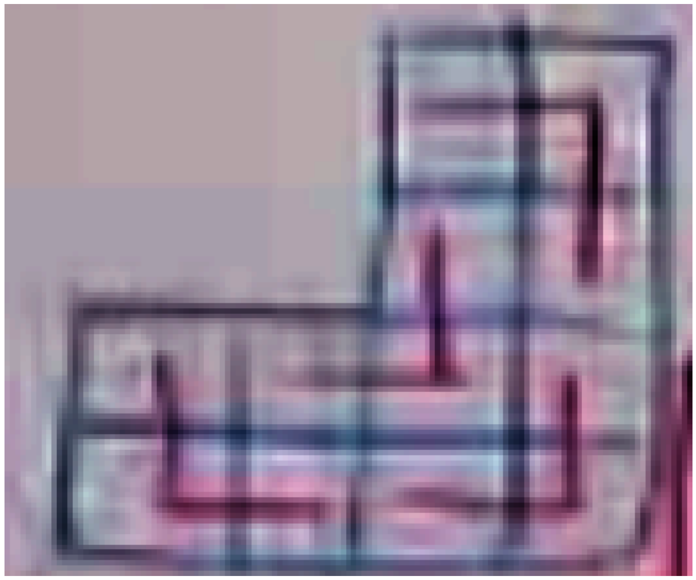


Figure 4. Alexander illustrates his base case.

Alexander: Where, uh, uh, n equals 2. Because it's 2 n minus one, so you get one, and zero's not a natural number. So these are all, one, uh, these are all one, sq, these are all one by one (points to the individual squares in the picture). The whole thing was this (gestures to the whole picture). Each of this is one tile (Points to individual squares). And since we're filling in with these looking things (points to another picture of three large blocks in the shape of an L), then you can do it with the n equals 2 case. (John: k) And then, for the, so assume you can do it for the, you can make these tiles for the n case. And then for the n plus one case, (draws a new picture, of three blocks in the shape of an L) where, where these are all n by n (Labels each of the three blocks "n x"),

John: You mean 2 to the n by 2 to the n?

Alexander: Yeah. (Revises some labels) Uh, (Subdivides the picture to look like Figure 5)

Figure 5. Alexander's n+1 case.

Uh, This is a smaller version (points to the three small blocks on the inside corner of the L). So this is. So these tiles, like, this red one right here, so we're taking this red one (draws an arrow and another red L, see Figure 6), is, each of these is 2 to n minus one by 2 to the n minus one. So, we know we can do that by, uh, hypothesis. And then, so we can do each of those (Draws in red lines as he says "each of those" see Figure 7).

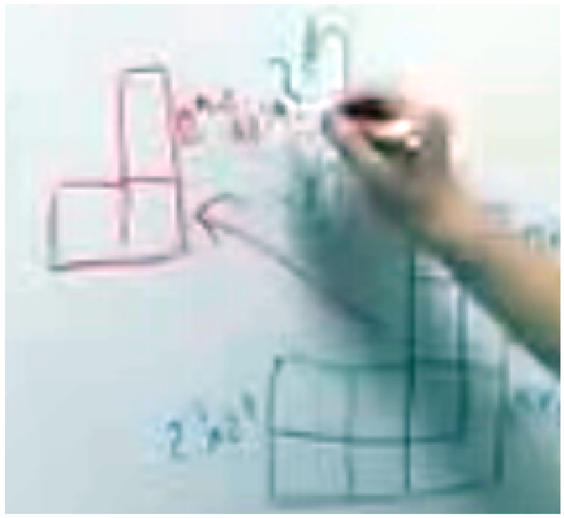


Figure 6. Alexander demonstrates where the n case is found in the n+ 1 case.

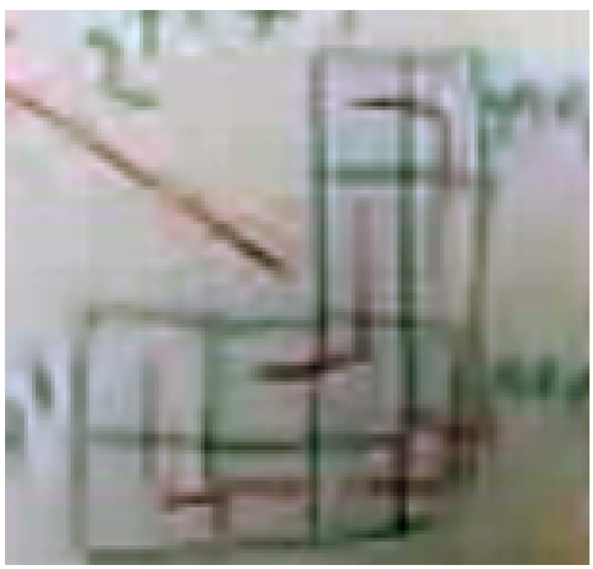


Figure 7. Alexander shows 4 "n" cases.

Alexander found a clever proof of the chessboard problem that involved two induction arguments. As with Mark, the *triggering event* was asking him to clarify or make more precise a step in the argument. However, in this example it is not clear that Alexander is learning something new about induction in general, but rather only clarifying his argument.

However, this does not mean that Alexander's conception of induction did not change throughout the study. In the initial interview, he described induction as the least convincing form of proof. In the final interview, I asked him to view that clip and I asked him if his thinking had changed. While watching this clip he smiled and laughed, and at the end, without hesitation, he said they are convincing. I asked him why, and he said he had seen a proof, in class, based on successor functions. Again, I think what he referred to is the discussion on Peano's axioms. While he said he did not think about the ideas between the interviews, he said in his deductive logic class he worked with successor functions and had to prove things about them. So he had thought about the ideas, just in another context. The *trigger* here was his deductive logic class. However, the class did not talk about induction specifically. This suggests it was only a *trigger* for Alexander because of his *personal reflections about mathematics*. At the very least, Alexander took an active role in making the connections.

Discussion. For Alexander, what stimulated the most significant change in his understanding of induction was his personal questions about induction, in particular its formalization. Since he was exposed to successor functions in his deductive logic class, he thought about these functions more than he would have otherwise. This *reflection about mathematics* allowed him to resolve his personal questions. Alexander was not convinced until he felt comfortable with the formalization. He was the one who pushed himself to make those connections.

The research techniques employed in this study, unfortunately, did not allow me to see what went on in his deductive logic class, or for that matter, any events outside the classroom and project that may have acted as triggers.

Chapter 7: Conclusions

While each of the students in the study had different experiences and grew in different ways, the thematic axes (*triggering events, personal questions about mathematics,* and *personal questions about a particular solution*) highlighted patterns in the narratives and from these patterns a theoretical perspective emerged. Reflection, both on mathematics in general and about specific problems, was central to each student's growth. The personal reflections of students and triggering events influenced each other in the following way. The questions students wondered about impacted which trigger might elicit growth, while triggers caused students to rethink assumptions and reflect on mathematics or specific problems. The reflections through which triggers led to growth, along with the reflections that were subsequent results of triggering events can be understood to constitute an *investigative orientation*. Each narrative reflects a different investigative orientation, motivated by different personal needs as well as different triggering events. Significantly, each investigative orientation affected what *kind* of knowledge was constructed.

Part of Mark's investigative orientation was his tendency to question his own proofs. As Mark reflected in the moment of actually proving something, he was able to revise his proof to make it clearer. This helped him understand induction in general better, as well as, the problem he was solving. Mark's tendency to reflect on the validity of his solution provided fertile ground for triggers that take the form of questions that ask for greater precision. In Mark's case, his personal questions influenced what triggers stimulated growth and he was able to construct new knowledge that allowed him to provide a precise solution to the problem at hand. While this knowledge will be useful to Mark in solving many induction problems, it was built in the context of solving a particular problem because his personal reflections were in the context of particular problems.

Sarah expanded her conception of what induction could apply to by her exposure to rich, complex problems, which went beyond symbol manipulation. While Sarah's initial personal questions did not come up in the interviews, it is clear that the tasks she engaged in raised questions for her. Her questions centered on when induction could be used, her conception of which expanded because of the tasks she participated in. In Sarah's case, the trigger affected what reflections took place and resulted in a different kind of knowledge construction, knowledge *about* induction instead of knowledge *of* induction.

Alexander also had the experience of revising his thinking in the moment to help clarify his argument. Alexander also had a desire to understand induction at a formal, axiomatic level. He felt more comfortable with the formalization of induction as he thought about topics from his logic class. Thus, experiences and reflection outside the mathematics classroom were important to his growth. Alexander wanted to formalize induction to answer his personal questions about what induction is, why it works, and how to avoid mistakes. This allowed him take experiences from a class not directly related to mathematical induction and learn from them. Alexander's situation is more similar to Mark's than to Sarah's in that his personal questions affected what events triggered growth. However, it is different from Mark's in that Alexander's personal questions had to do with the formal underpinnings of induction and so Alexander constructed knowledge related to these formalities.

Even though each student was exposed to the same stimuli, i.e. the same classroom process, the same text, and the same problems assigned, each student built a different kind of knowledge, in part because their personal backgrounds, and the inquiries that were built from these specific backgrounds, contrasted sharply. Such variation suggests strongly that because students can have strikingly different experiences, even though they have been placed in the same learning situation, instruction could benefit by taking such differences into account, and helping students build from them. One way this could happen is the instructor could allow students to pursue their worthwhile personal questions.

Chapter 8: Discussion

The results so far have several potential implications for researchers and teachers. First, the analysis above suggests theoretical perspectives that could, at least to some extent, suggest a unifying context for contrasting findings of some prior studies. Avital and Hansen (1976) gave suggestions for how to teach induction that focused on student exploration. Wistedt and Brattström (2005) emphasized the importance of the teacher in the learning process and said that student exploration was not enough. The findings here suggest ways to make sense of this seeming disparity. As we have seen above, exploration allows students to become more sensitive to their own investigative orientations, and hence direct personal inquiry to areas where they might be ready to build further understanding. However, each student, as an individual, may not know which questions to ask to trigger growth for other students.

Second, the findings here invite us to reconsider several widely held assumptions. We might be cautious about studies that involve stage theories whose underlying frameworks make generalizations across student populations without considering significant, indeed fundamental, individual differences. Different students, as we've seen, may not go through the same developmental stages, or respond to given tasks in the same way. For example, Harel (2008, pg 122) distinguishes between types of problems that require the solver to assume the hypothesis for an object, versus those that require the assumption for a whole class of objects, but the analysis of Mark above suggests that Harel's framework needs to be refined. To be specific, Harel's distinction carries an unstated prior assumption: that the type of problem is an attribute of the problem, independent of the student working on that problem. For Mark, however, proving generalized De Morgan's law was not a problem. To prove a generalized De Morgan's law, the solver needs to make, and then build from, an assumption about "sets." It is unclear to me if

Harel would consider Generalized De Morgan's law a problem of the first or second type, i.e. if the solver has to make an assumption about an object or about a class of objects. Harel's classification depends on whether a set is understood to be an object or a class of objects. This distinction is impossible to make without considering the student solving the problem. If a set is an object for the student, it is a problem of the first type. If a set is a class of objects for the student, it is a problem of the second type. That one cannot classify such problems without considering the student working on the problem, however, does not invalidate prior research such as Harel's. It is certainly possible that stages of the kind Harel has emphasized might appear as trends, in the aggregate, across samples of many students. Even so, on the bases of the cases studied here, one should be careful not to make assumptions based on aggregate behavior about how particular students may approach a given problem. Also, it may well be best to let, or, better still, encourage students to pursue personal lines of inquiry, to explore topics that they have personal questions about, and so keep instruction personal. In particular, based on the findings here, I believe we have good reason to examine critically the widely held idea that there exists a single optimum curriculum or classroom methodology that can lead students to build ideas in a specific way.

Second, the work reported here suggests possibilities for further investigation. The present conclusions build from a particular interpretive approach, derived from close analysis of work by a small student sample. Hence not just the findings here, but also the interpretive approach they build from, could be tested with a larger student sample and appropriately adapted methods. One might attempt, for example, with a larger student sample, to locate widespread, statistically significant trends or patterns across different students' investigative orientations, hence to explore to what extent each student's orientation is completely personal or unique.

Another place to look for trends or patterns might be in how students respond to given triggers. Some might be more effective than others, through the ways that they elicit growth across a variety of personal investigative orientations.

With the students in this study, growth occurred when they reflected, often during research interviews in which they were invited to explain their thinking and its motivations. A teacher can encourage such reflection by pushing students to make arguments more precise, or by posing considered, sympathetic questions about why a particular technique or procedure works. Hopefully, this may not just invite students to reflect on the particular problem at hand, but will also help them to develop or refine existing habits of reflection.

Appendix A: Questionnaire

Questionnaire

Name:

Year at BYU:

What prior experience have you had with mathematical induction, if any?

Briefly describe your understanding of mathematical induction:

Appendix B: Full Transcript of Sarah's Event

Sarah (video): 'Cause like, for impact, we did, like I would a proof and it would show, like I didn't get why, like, you assume it's true for, like, one or something, and then you do it for, so you assume it's true for some random thing, and then you show that the next one would be true, and that didn't make sense to me how that proved why everything was then true, like the whole sequence or whatever you're proving. So that didn't make sense to me. But then doing it in class, I guess, when he just started doing more examples, is showing more that it was a sequence or something that we were proving or something like that. Like you'd show it was true for one. Then for some random k, and then for the next one. That made more sense to me, 'cause you could just use it on the base case. Show the next one and then the next one and keep going. That's what made it click for me.

I ask her if her understanding is different

Sarah: Okay. Umm…I think it's kind of the same. Except now, like, because of the problems that we did when we all worked together (John: uh-huh), umm, I've seen just more how it can be used (ok), and that makes me understand it better still, I think. 'Cause before I just thought that, I only, like, understood how you would apply it to like a series really I guess. But now, having, having seen it done with other examples, I see how you break it down and then kind of, would have to show by induction, like, with like a series, or something, kind of. I don't know if that was clear.

John: What do you mean by a series or something?

Sarah: So, it's not, you wouldn't necessarily break down the problems that we did into series, or like, but it's, it's a pattern that you would show, (John: ok) and by like the pattern is how you would show, like, you can show it's true by induction.

John: Ok, How does that differ from…

Sarah: From what I thought before? Like before, I always, was stuck, in like, like, uh, like, express it mathematically. Like I, I didn't know how to do it not with just, I don't know, like, hmm…

John: What do you mean by express it mathematically?

Sarah: So, like I would write it all out in symbols. And be like this is the base case and this is the next case, and this the next case, and this is the next case. (ok) But, and I didn't think that you could generalize it to be like, well, think of this as your base case. (uh-huh) Like, I thought the base case would be like for n equals 1 or for the first whatever the thing was defined. I guess it's still is kinda the same, I just hadn't seen it been applied to like more of a pattern than like some kind of series in mathematics.

John: When you say series do you mean like (Sarah: By series I mean like) a summation?

Sarah: Yeah, you can, you can express this as, this like series as a summation or something, and then you prove that that expressed what you had before, by induction.

John: Ok

Sarah: I don't know if that makes sense either.

John: Umm, well, I mean I can guess what you're saying, I guess

Sarah: Kind of

John: So maybe.

Sarah: Like, for the most part it's the same.

John: Ok

Sarah: It's, It's just like (John: And in what ways…) I can see more how's it's been applied, like it's becoming more clear, like I see now more the ways you can use it.

John: Ok, so I guess my question is, umm, in what ways is it the same for you and in, and in what ways, like, umm, you've mentioned that it was like a different, I don't kind of a scenario maybe? (yeah) Is that a fair way? (yeah) So, in like, in what ways is the chessboard problem different from the types of problems that you've done before?

Sarah: Ok. So, well the chessboard problem was more, like, I guess I just mean you can't, the chessboard problem you couldn't just write out, well like, having like a two to the n by a two to the n and then you just plug in like the different numbers, it wasn't, like less, it's not even computational, the way I did it before, but more just, you can write down a definite, like this and then this and then this, whereas you'd have to use more words, and explain with the chessboard problem, or something, or show with pictures that it works, at least that's the only, that's what I got from what we did on Saturday.

John: Ok, good.

Sarah: And so just seeing that like induction, 'Cause I hadn't, I'd only seen it applied, to just when you would just express it in numbers and symbols. So I hadn't seen like how you'd use it be like, uhh.., I don't think I'm making more sense still

John: No, I do think, I, I think you, you

Sarah: But…

John: Go ahead

Sarah: I don't know

John: So I guess, umm, what I'm hearing you say, tell me if I'm right or wrong (ok), is that umm, before there were, there was, umm, maybe some sort of formula that (yeah) you (right) were trying to prove (yeah). So you could kind of uhh..

Sarah: You just (mess…) you knew what you were arriving at and you knew what started with and you just kind of get there. Like, it was just systematic like plug this in do this swap this around and then you could get there or something.

John: Yeah, (Whereas) Go ahead

Sarah: Whereas, now it's just you have more, you want to see if something is true and so you're, ummm.., It's like not a formula you're getting to, but you're just showing it would be true in any case.

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